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TRANSFORMATION OF THE GIL'DEN–MESHCHERSKII PROBLEM TO ITS STATIONARY FORM AND THE LAWS OF MASS CHANGE*

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The autonomization method is applied to one of the possible formulations of the Gil'den—Meshcherskii problem of two bodies, representing two points of variable mass. In order to find all possible mathematical laws of mass change which have to be fulfilled, the equation of motion is to be reduced to its stationary form. It is found that the real mass can be neither periodic, nor oscillatory. The laws established include, as particular cases, the classical laws of Meshcherskii and Eddington—Jeans. All rectilinear solutions of the Gil'den—Meshcherskii problem are also obtained.

1. One of the best known problems of celestial mechanics is the classical, nonstationary Gil'den—Meshcherskii problem which is used to describe the evolution of double stars with secular loss of mass caused by the photonic and corpuscular activity. The Gil'den—Meshcherskii problem serves as a mathematical model for describing various cases of motion of the bodies of variable mass such as: the motion of a mass point in the gravitational field of a body with a variable mass; or a relative motion of two bodies of variable mass in the case when their Newtonian interaction exceeds appreciably the reaction forces; also when a perturbing force, frictional in character and compensating the reaction forces are present, etc. (see e.g. /1—3/).

We consider the equation of motion of the form

$$\mathbf{r}^{\cdot\cdot} = -\mu(t)\,\mathbf{r}/r^3 \tag{1.1}$$

Here $\mathbf{r} = (x, y)$ is the radius vector of the motion of one mass point relative to other point in the orbital plane, $\mu(t)$ is a function of time t, and $r = |\mathbf{r}|$. We note that the application to the two-body problem of the Dirac's cosmological hypothesis /4/ postulating that the gravitational constant changes with time, yields the same equation.

The following laws of mass change $\mu(t)$ under which the equation (1.1) can be integrated in quadratures, called respectively the first, second and the unified Meshcherskii laws, are known:

$$\mu(t) = (\alpha t + \beta)^{-1}, \quad \mu(t) = (\alpha t + \beta)^{-1/2}, \qquad \mu(t) = (\alpha t^2 + \beta t + \gamma)^{-1/2} \tag{1.2}$$

The laws (1.2) can be physically substantiated with help of the Eddington—Jeans theory /5/ of internal structure and evolution of stars

$$\mu^{\cdot} = -k\mu^{\nu} \tag{1.3}$$

where k is a proportionality coefficient and the index ν satisfies the inequality $1 < \nu \leqslant 3$ ($\nu = 2$ gives the first, and $\nu = 3$ the second Meshcherskii law). The relation (1.3) can, of course, be generalized by removing the restrictions imposed on the index ν (in particular, when $\nu = 0$, the mass varies linearly, while with $\nu = 1$ it varies exponentially).

Using the variable transformation

$$\mathbf{r} = v(t)\,\boldsymbol{\rho}, \quad d\tau = u(t)\,dt, \quad \boldsymbol{\rho} = (\xi, \eta), \qquad v(t) \in C_I^2, \quad u(t) \in C_I^2, \quad u(t)\,v(t) \neq 0, \quad \forall\, t \in I \tag{1.4}$$

where I is an open, bounded or unbounded interval on the time t-axis and C_I^2 is a space of functions twice continuously differentiable on I, we can reduce the problem (1.1), using the mass change laws given above, to the stationary form

$$\boldsymbol{\rho}'' \pm b_1\boldsymbol{\rho}' + b_0\boldsymbol{\rho} = -\mu_0\boldsymbol{\rho}/\rho^3, \quad (') = d/d\tau \tag{1.5}$$

Here b_0, μ_0 are real constants and b_1 can either be a real, or a purely imaginary constant.

By virtue of the Stäckel—Lie theorem, (1.4) represents the most general transformation preserving the order of the equation, the linearity of its linear part, and the structure of its nonlinear part.

We shall establish all possible laws of mass change under which the problem (1.1) can be transformed, using (1.4), to the form (1.5).

Since the laws sought represent not only the sufficient, but also the necessary conditions of existence of the corresponding transformation (1.4), it means that the solution of our problem cannot be obtained by semi-inverse method, nor by the heuristic substitutions. The solution obtained is based on the method of autonomization of differential equations /6/.

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2. **Lemma 1.** The problem (1.1) will be transformed by means of (1.4) to the form (1.5) it is necessary and sufficient that the kernel $\mu(t)$ and the multiplier $v(t)$ of the transformation (1.4) satisfy the equations

$$\frac{1}{2}\frac{u^{\cdot\cdot}}{u} - \frac{3}{4}\left(\frac{u^{\cdot}}{u}\right)^2 - \frac{1}{4}\delta u^2 = 0, \quad \delta = b_1^2 - 4b_0 \tag{2.1}$$

$$v^{\cdot\cdot} - b_0 v^{-3} = 0, \quad b_1 = 0 \tag{2.2}$$

$$v^{\cdot\cdot} - \frac{b_0}{b_1^2} v^{-3}\left(\int_{t_0}^{t} v^{-2}\,dt\right)^{-2} = 0, \quad b_1 \neq 0, \quad t_0 \in I \tag{2.3}$$

Here $v(t)$, $u(t)$ and $u(t)$ are connected by the following relations:

$$\mu(t) = \mu_0 u^2(t) v^3(t), \quad v(t) = |u|^{-1/2}\exp\left(\pm\frac{1}{2} b_1 \int u\,dt\right), \quad v^{\cdot\cdot} - b_0 u^2(t)\, v = 0 \tag{2.4}$$

In addition, the problem (1.1) admits a one-parameter Lie group with the infinitesimal operator

$$X = \frac{1}{u}\frac{\partial}{\partial t} + \frac{v^{\cdot}}{uv}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) \tag{2.5}$$

and has particular solutions

$$\mathbf{r} = v(t)\boldsymbol{\lambda}, \quad \lambda^3 = -\mu_0 / b_0, \quad b_0 \neq 0 \tag{2.6}$$

We note that the Lemmas and theorems appearing in this paper are all given without proof.

Lemma 2. General solution of (2.1) has the form

$$u(t) = \begin{cases} (\alpha_1 t + \beta_1)^{-1}(\alpha_2 t + \beta_2)^{-1}, & \delta = (\alpha_1\beta_2 - \alpha_2\beta_1)^2 > 0 \\ (At^2 + Bt + C)^{-1}, & \delta = B^2 - 4AC < 0 \\ (\alpha t + \beta)^{-2}, & \delta = 0 \end{cases}$$

Particular cases of this solution are described by the formulas

$$u(t) = (\alpha t + \beta)^{-1}, \quad u(t) = 1$$

Lemma 3. a) General solution of (2.2) has the form

$$v(t) = [(\alpha_1 t + \beta_1)(\alpha_2 t + \beta_2)]^{1/2}, \quad -4b_0 = (\alpha_1\beta_2 - \alpha_2\beta_1)^2 > 0$$
$$v(t) = (At^2 + Bt + C)^{1/2}, \quad -4b_0 = B^2 - 4AC < 0$$
$$v(t) = \alpha t + \beta, \quad b_0 = 0$$

and important particular cases of this solution are

$$v(t) = \sqrt{\alpha t + \beta}, \quad v(t) = 1$$

b) General solution of (2.3) is given by the relations

$$v(t) = (\alpha_1 t + \beta_1)^{\gamma_\pm}(\alpha_2 t + \beta_2)^{\gamma_\mp}, \quad \gamma_\pm = 1/2 \pm b_1/(2\sqrt{\delta}), \quad \delta = (\alpha_1\beta_2 - \alpha_2\beta_1)^2 > 0$$

$$v(t) = (At^2 + Bt + C)^{1/2}\exp\left(\pm\frac{b_1}{\sqrt{-\delta}}\operatorname{arctg}\frac{2At + B}{\sqrt{-\delta}}\right), \; \delta = B^2 - 4AC < 0, \quad v(t) = (\alpha t + \beta)\exp[\mp b_1 / (2\alpha(\alpha t + \beta))], \quad \delta = 0$$

and particular cases of this solution have the form

$$v(t) = (\alpha t + \beta)^{1/2 \pm b_1/(2\alpha)}, \quad v(t) = \exp(\pm 1/2\, b_1 t)$$

3. **Theorem 1.** Problem (1.1) will be transformed to the form (1.5) with $b_1 = 0$, it is

$$\mathbf{r} = |u|^{-1/2}\boldsymbol{\rho}, \quad d\tau = u\,dt \tag{3.1}$$

necessary and sufficient that the mass $\mu(t)$ satisfies the differential equation

$$\mu^{\cdot\cdot} - 2\mu^{-1}\mu^{\cdot 2} + b_0\mu^5 = 0 \tag{3.2}$$

Here the relations (3.1), (2.5) and (2.6) assume, respectively, the form

$$\mathbf{r} = \mu^{-1}\boldsymbol{\rho}, \quad d\tau = \mu^2 dt, \qquad X = \mu^{-2}\frac{\partial}{\partial t} - \mu^{-3}\mu^{\cdot}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right), \qquad \mathbf{r} = \mu^{-1}\boldsymbol{\lambda}, \quad \lambda^3 = -\mu_0 / b_0, \quad b_0 \neq 0$$

Theorem 2. The transformation

$$\mathbf{r} = |u|^{-1/2}\exp\left(\pm\frac{1}{2} b_1 \int u\,dt\right)\boldsymbol{\rho}, \quad d\tau = u\,dt \tag{3.3}$$

transforms the problem (1.1) to the form (1.5) $(b_1 \neq 0)$ it is necessary and sufficient so that the mass $\mu(t)$ satisfies the integrodifferential equation

$$\mu^{\cdot\cdot} - 2\mu^{-1}\mu^{\cdot 2} + \frac{9b_1^2 - \delta}{36 b_1^2}\mu^5\left(\int_{t_0}^{t}\mu^2\,dt\right)^{-2} = 0 \tag{3.4}$$

The relations (1.4), (2.5) and (2.6) assume , in this case, the form

$$\mathbf{r} = \left(3b_1 \int_{t_0}^{t} \mu^2\, dt\right)^{2/3} \mu^{-1}\boldsymbol{\rho}, \quad d\tau = \left(\pm 3b_1 \int_{t_0}^{t} \mu^2\, dt\right)^{-1} \mu^2\, dt$$

$$X = \left(\pm 3b_1 \int_{t_0}^{t} \mu^2\, dt\right) \mu^{-2} \frac{\partial}{\partial t} + \left[\left(\mp 3b_1 \int_{t_0}^{t} \mu^2\, dt\right)^{-1} \mu^{\cdot} \pm 2b_1\right] \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}\right)$$

$$\mathbf{r} = \left(3b_1 \int \mu^2\, dt\right)^{2/3} \mu^{-1}\boldsymbol{\lambda}, \quad \lambda^3 = -\frac{\mu_0}{b_0}, \quad b_0 \neq 0$$

respectively.

We shall call the equations (3.2) and (3.4) the differential and integrodifferential laws of mass change, respectively. Finite formulas can be obtained either by direct integration of the equations (3.2) and (3.4), or by employing the relation connecting $\mu(t)$ with $u(t)$ and $v(t)$ (formula (2.4), lemmas 2 and 3).

4. **Theorem 3.** All possible laws governing the change of mass $\mu(t)$ with time in the problem (1.1), (1.4), (1.5), are given by the following finite equations:

$$\mu(t) = (\alpha_1 t + \beta_1)^{\varkappa_\pm} (\alpha_2 t + \beta_2)^{\varkappa_\mp}, \quad \varkappa_\pm = -1/2 \pm 3b_1/(2\sqrt{\delta}), \quad \delta = b_1^2 - 4b_0 = (\alpha_1\beta_2 - \alpha_2\beta_1)^2 > 0, \ \alpha_1 \neq 0, \ \alpha_2 \neq 0 \tag{4.1}$$

$$\mu(t) = (At^2 + Bt + C)^{-1/2} \exp\left(\pm \frac{3b_1}{\sqrt{-\delta}} \operatorname{arctg} \frac{2At + B}{\sqrt{-\delta}}\right), \ \delta = B^2 - 4AC < 0 \tag{4.2}$$

$$\mu(t) = (\alpha t + \beta)^{-1} \exp\left[\mp \frac{3b_1}{2\alpha(\alpha t + \beta)}\right], \quad \delta = 0 \tag{4.3}$$

$$\mu(t) = (\alpha t + \beta)^{-1/2 \pm 3b_1/(2\alpha)} \tag{4.4}$$

$$\mu(t) = \mu_0 \exp(\pm 3/2\, b_1 t) \tag{4.5}$$

Corollary. The real mass $\mu(t)$ in the problem (1.1), (1.4), (1.5) can vary neither periodically, nor oscillatorily.

Note. The Meshcherskii laws follow from the formulas (4.1)—(4.4) with $b_1 = 0$, and the formulas (4.4), (4.5) represent a finite form of the generalized Eddington—Jeans law. The formulas

$$\mu(t) = \alpha t + \beta \tag{4.6}$$

$$\mu(t) = (\alpha t + \beta)^{-2} \tag{4.7}$$

$$\mu(t) = (\alpha_1 t + \beta_1)(\alpha_2 t + \beta_2)^{-2} \tag{4.8}$$

are particular cases of (4.4). The linear law (4.6) was studied by Lowett (see /1/), who however mistakenly assumed the first Meshcherskii law to be incorrect. The authors of /7/ established that the cases (4.6) and (4.7) can be used to study the motion in a medium with resistance. In /8/ it was shown how, using the known initial conditions, one can establish the phenomena of capture or decomposition of a system, determined by the change of mass, in the past as well as in the future. The relations (4.8) /9/ contain the formula $\mu(t) = (2\alpha t + 1)(\alpha t + 1)^{-2}$ obtained in /10/ in the course of investigating a stationary problem of motion of a (point) satellite in the earth's (sphere) gravity field, acted upon by the resistance of a homogeneous atmosphere proportional to the velocity. The formula (4.1) was obtained by L. M. Berkovich and B. E. Gel'fgat /11/ (*).

Theorem 4. In order that the problem (1.1) would admit the rectilinear solutions (2.6) there is necessary and sufficient if the mass $\mu(t)$ varies according to the laws (4.1)—(4.5). Here the relations defined by Lemma 3 ($b_0 \neq 0$) are used as $v(t)$.

In conclusion we note that the problem of the mathematical laws of mass change are closely linked to the problem of integrating (1.1), and the latter problem requires special attention.

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*) B. E. Gel'fgat (1929—1976) who made considerable contribution to the study of the Two-body problem with variable mass, died tragically in July 1976 while making an ascent of one of the Pamir peaks.

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